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T E EXPONENTIAL DISTRIBUTION AND ITS ROLE IN LIFE TESTING

BY

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1. Introduction

Many current results in life testing are based on the assumption that the life X is described by a probability density function $f(x;\theta)$ of the form

$$(1) \qquad f(x;\theta) = \frac{1}{\theta} \exp(-x/\theta) \quad , \quad x > 0 \quad , \quad \theta > 0 \quad .$$

In (1), x is life measured in appropriate units (for example, hours) and

$$(2) \qquad E(X) = \int_0^\infty x f(x;\theta)\, dx = \int_0^\infty \frac{x}{\theta} \exp(-x/\theta)\, dx = \theta$$

is the mean life expressed in appropriate units. There is evidence that the lives of electron tubes or the time intervals between successive breakdowns of electronic systems are, to a first approximation, random variables having the density (1).

A partial justification for the assumption of an exponential p.d.f. has been discussed in some detail in a paper by the author [12, 13] and by D. J. Davis [7] and several relevant references may be found in these papers. Further evidence of an empirical nature can be found in a recent series of ARINC monographs [1]. We are well aware of the fact that many life distributions are not adequately described by equation (1). While this may be the case, an understanding of the theory in the exponential

case is essential if we are to treat more general situations.

As we write these words, some six years have elapsed since we started research on statistical methods in life testing. At the beginning some basic questions arose as to where to distribute our efforts since there were many avenues open for research activity. The exponential distribution was chosen only after considerable discussion with people in the field of electronics and after a study of the literature existing and available at that time. In retrospect the choice of the exponential distribution was a good one. It seems as if the exponential distribution plays a role in life testing analogous to that of the normal distribution in other areas of statistics. It is our feeling that in many cases there is at least as much justification for using the exponential distribution in life test situations as to use the normal distribution, for example, in developing sampling plans by variables.

An important by-product of the assumption of the exponential distribution of life is that it makes it possible to apply the well developed theory of Poisson processes. Furthermore, one can by almost trivial changes generalize all the results to the case where the conditional rate of failure is some function of time, $Z(t)$, rather than a constant as in the exponential case. The theory thus extended has validity over a wide area including most cases of practical interest.

2. Poisson Processes and Exponential Models [1]

We now consider in some detail why one might expect the exponential distribution to occur and what implications the assumption of an exponential

[1] For a detailed treatment of Poisson processes and exponential distributions see Feller [14].

distribution carries with it. In addition we will mention why one might expect certain other distributions to also occur in life test situations.

One often refers to the exponential distribution as corresponding to a purely random failure pattern. Precisely what one means mathematically is that whatever is causing the failure occurs according to a Poisson process with some rate, λ. For example, if we imagine that a failure occurs whenever a Geiger counter is actuated by a radioactive source having an emission rate, λ, then the distribution of time intervals between successive failures will be given by the p.d.f. $\lambda e^{-\lambda t}$, $\lambda > 0$, $t > 0$. This is very easy to prove. Let T be the random variable associated with the time interval between successive events, then

(3) $\Pr(T > t) = \Pr$ [no event occurs in the interval $(0,t)$] , where $t = 0$ is the time when the most recent event occurred.

From the Poisson assumption,

$$\Pr(T > t) = e^{-\lambda t} \quad . \tag{4}$$

Thus

$$\Pr(T \leq t) = 1 - e^{-\lambda t} \tag{5}$$

and the p.d.f. is given by

$$f(t) = \lambda e^{-\lambda t} \; , \; \lambda > 0 \; , \; t > 0 \quad . \tag{6}$$

The question naturally arises as to whether this rather artificial model has any relevance to a real life situation. The answer is that it may under the following sort of conditions: Imagine a situation where

a device under test is being subjected to an environment E , which is some sort of random process. Let us imagine that this random process has peaks distributed in a Poisson manner and that it is only these peaks that can affect the device, in the sense that the device will fail if a peak occurs and will not fail otherwise. If this is the situation and if peaks in the stochastic process describing the environment occur with Poisson rate, λ , then the failure distribution for the devices under test will be given by the p.d.f. (6). It is interesting to note that while we call (6) a failure distribution it describes, in reality, the frequency of severe shocks in the environment. This is precisely what (6) means in the all or none situation, where the device fails if and only if a peak occurs and not otherwise.

It is not necessary, in the preceding discussion, that we have an all or none situation in order that the exponential distribution arise. Suppose, as before, that peaks in the stochastic process occur with rate λ and that the conditional probability that the device fails given that a peak has occurred is p . Then it is clear that the event: "device does not fail in time t " is composed of the mutually exclusive events, "no peak occurs in $(0,t)$ ", "one peak occurs and device does not fail given this peak," "two peaks occur and device does not fail given these two peaks," etc. Symbolically we have

$$(7) \qquad \begin{aligned} Pr(T > t) &= e^{-\lambda t} + q(\lambda t)e^{-\lambda t} \\ &\quad + q^2 \frac{(\lambda t)^2}{2!} e^{-\lambda t} + \dots + q^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} + \dots \\ &= e^{-\lambda t}[1+q\lambda t + \frac{(q\lambda t)^2}{2!} + \dots + \frac{(q\lambda t)^k}{k!} + \dots] \\ &= e^{-\lambda t+\lambda q t} = e^{-\lambda(1-q)t} = e^{-\lambda p t} \end{aligned}$$

Thus

(8) $$\Pr(T \leq t) = 1-e^{-\lambda p t}$$

and

(9) $$f(t) = \lambda p e^{-\lambda p t} , \ t > 0 \ .$$

Again we have an exponential distribution. It is interesting to note that the exponent λp (which is the mean time between failures) reflects the simplest kind of interaction between the device being tested and its environment. If we had two kinds of devices D_1 and D_2 each subjected to the same environment E, and if p_1 is the conditional probability that D_1 will fail given that a peak in E has occurred, and p_2 is the conditional probability that D_2 will fail given that a peak in E occurs, then the mean time between failures is given by λp_1 and λp_2 respectively. If $\lambda p_1 < \lambda p_2$ then one can say that in environment E, the device D_1 is stronger than the device D_2.

Carrying these ideas further, suppose that an item is exposed simultaneously to k environments $E_1, E_2, \ldots, E_k$. Suppose that the environment E_i is characterized by a rate λ_i (the frequency with which dangerous peaks occur in environment E_i) and that the conditional probability that the device fails given that a peak has occured is p_i. From (7) it follows that the probability that a device survives environment E_i for a length of time t is given by $e^{-\lambda_i p_i t}$. Let us assume that the environments are described by stochastic processes which operate independently of one another, then from (7) and the assumption of independence, the time T

until a failure occurs (or the time between failures) is a random variable such that

$$(10) \qquad Pr(T > t) = \prod_{i=1}^{s} e^{-\lambda_i p_i t} = e^{-(\sum_{i=1}^{s} \lambda_i p_i)t}$$

Defining $\Lambda = \sum_{i=1}^{s} \lambda_i p_i$, (10) becomes

$$(11) \qquad Pr(T > t) = e^{-\Lambda t}$$

and consequently

$$(12) \qquad Pr(T \leq t) = 1-e^{-\Lambda t} , \; t \geq 0 \; .$$

Thus we again have an exponential distribution.

In the situation described by the distribution function (8) and density function (9), p, the conditional probability that a failure occurs given that a peak has occurred, is independent of t . Let us now assume that given that a peak occurs, the conditional probability that a failure occurs is given by $p(t)$. It is easy to show in this case that

$$(13) \qquad Pr(T > t) = e^{-\lambda\int_0^t p(\tau)d\tau}$$

and so

$$(14) \qquad Pr(T \leq t) = 1-e^{-\lambda\int_0^t p(\tau)d\tau}$$

and

$$f(t) = \lambda p(t)e^{-\lambda\int_0^t p(\tau)d\tau} = \lambda p(t)e^{-\lambda P(t)},\ t \geq 0 \tag{15}$$

where

$$P(t) = \int_0^t p(\tau)d\tau \quad . \tag{16}$$

Illustrations

1. In particular suppose that $p(\tau) = 0,\ 0 \leq \tau < A$ and $p(\tau) = 1,\ \tau \geq A$, then

$$\Pr(T \leq t) = 0,\ t < A$$

and

$$\Pr(T \leq t) = 1-e^{-\lambda(t-A)}\ ,\ t \geq A$$

and

$$f(t) = 0\ ,\ t < A$$

and

$$f(t) = \lambda e^{-\lambda(t-A)}\ ,\ t \geq A\ .$$

This is known as the two parameter exponential.

2. Another example is of the following kind:

$$p(\tau) = \left(\frac{\tau}{t_0}\right)^{\alpha},\ 0 \leq \tau < t_0$$

$$= 1,\ \tau \geq t_0\ .$$

In this case

$$\Pr(T \leq t) = 1-e^{-\frac{\lambda t^{\alpha+1}}{(\alpha+1)\, t_0^{\alpha}}}, \quad 0 \leq t < t_0$$

$$= 1-e^{-\lambda(t-\frac{\alpha}{\alpha+1} t_0)}, \quad t > t_0 \quad .$$

Remark: In the range, $0 \leq t < t_0$, $\Pr(T \leq t)$ is of the form frequently called a Weibull distribution.

A generalization in still another direction is to assume that devices are exposed to a variety of environments. For example, we could imagine a situation where devices are exposed to k possible environments $E_1, E_2, \ldots, E_k$ which can occur with respective probabilities $c_1, c_2, \ldots, c_k (c_j \geq 0, \sum_{j=1}^{k} c_j = 1)$. Furthermore, with each environment E_j there is associated a λ_j (the rate at which dangerous peaks occur in environment E_j) and within the environment E_j, the conditional probability that a device fails given that a peak occurs is p_j. It then follows quite readily that the probability of surviving for a length of time t is given by

$$(17) \qquad \Pr(T > t) = \sum_{j=1}^{k} c_j e^{-\lambda_j p_j t} \quad .$$

The associated density function is described by

$$(18) \qquad f(t) = \sum_{j=1}^{k} c_j \lambda_j p_j e^{-\lambda_j p_j t} = \sum_{j=1}^{k} c_j \nu_j e^{-\nu_j t} \quad ,$$

where

$$\nu_j = \lambda_j p_j \; .$$

Thus we are led to a description of the density of failure times as a sum of exponentials. A continuous analogue of (18) is

$$f(t) = \int_0^{\infty} \nu e^{-\nu t} \, dG(\nu) \quad . \tag{19}$$

It is interesting to see what happens in the special case for which

$$g(\nu) = \frac{dG(\nu)}{d\nu} = \frac{\nu^r A^{r+1} e^{-A\nu}}{r!} \, , \; 0 \leq \nu < \infty . \tag{20}$$

Equation (19) then becomes

$$f(t) = \int_0^{\infty} \frac{\nu^{r+1} \, A^{r+1} \, e^{-\nu(A+t)} \, d\nu}{r!} \quad . \tag{21}$$

It is readily verified that $f(t)$ becomes

$$f(t) = \frac{(r+1) \, A^{r+1}}{(A+t)^{r+2}} \quad , \quad t > 0 \quad . \tag{22}$$

Remark: It is interesting to note that in the special case where $A = r+1$, $f(t)$ becomes an $F(2, 2r + 2)$ distribution.

Thus, the assumption that failure is associated in an essential way with the occurrence of peaks in a Poisson process has led us to a number of quite interesting life distributions.

3. Model Based on Conditional Probability of Failure

Let us now proceed in a different direction and see where this leads us. It is well-known that if the underlying distribution of life T is

described by the exponential p.d.f. $f(t;\theta) = \frac{1}{\theta} e^{-t/\theta}$, then the conditional probability of an item failing in the time interval $(t, t + \Delta t)$ given that it has survived for time t , is independent of t . This is readily verified since

$$(23) \qquad Pr(t < T < t + \Delta t | T > t) = f(t;\theta)\Delta t \; / \; 1-F(t;\theta)$$

$$= \frac{1}{\theta} e^{-t/\theta} \Delta t / e^{-t/\theta} = \Delta t/\theta \quad .$$

Generally speaking the conditional rate of failure or hazard rate, $g(t)$, where

$$(24) \qquad g(t) = f(t)/1-F(t)$$

does depend on t . We are interested in solving for $f(t)$ and $F(t)$, given the failure rate, $g(t)$. This is readily done since (24) implies that

$$(25) \qquad d[\ln(1-F(t)] = -g(t)$$

and therefore, recalling that $F(0) = 0$,

$$(26) \qquad F(t) = 1 - e^{-\int_0^t g(\tau)d\tau} = 1 - e^{-G(t)} \; , \; t \geq 0 \quad ,$$

where $G(t) = \int_0^t g(\tau)d\tau$. The density function $f(t)$ is obtained by differentiation and is

$$(27) \qquad f(t) = g(t) \; e^{-\int_0^t g(\tau)d\tau} \; , \; t \geq 0 \quad .$$

Illustrations

Example 1. Suppose that $g(t) \equiv \frac{1}{\theta}$, $t \geq 0$, then

(26) becomes $$F_\theta(t) = 0 \ , \ t < 0$$
$$= 1 - e^{-t/\theta} \ , \ t \geq 0$$

and(27) becomes $$f_\theta(t) = \frac{1}{\theta} e^{-t/\theta} \ , \ t \geq 0 \ ,$$
$$= 0 \ , \text{ elsewhere.}$$

This is the exponential distribution.

Example 2. Suppose that $g(t) = 0$ for $0 \leq t < A$
$$= \frac{1}{\theta} \text{ for } t \geq A \ .$$

In this case we get the two-parameter exponential distribution where

(26) becomes $$F_\theta(t) = 0 \ , \ t < A$$
$$= 1 - e^{-(\frac{t-A}{\theta})} \ , \ t \geq A \ ,$$

and (27) becomes $$f_\theta(t) = 0 \ , \ t < A$$
$$= \frac{1}{\theta} e^{-(\frac{t-A}{\theta})} \ , \ t \geq A \ .$$

Example 3. Suppose that $g(t) = \frac{kt^{k-1}}{\theta^k}$, where $k > 0$. In this case,

(26) becomes $$F_\theta(t) = 0 \ , \ t < 0$$
$$= 1 - e^{-(\frac{t}{\theta})^k} \ , \ t \geq 0$$

and

(27) becomes $f_\theta(t) = \frac{kt^{k-1}}{\theta^k} e^{-(\frac{t}{\theta})^k}$, $t \geq 0$

$= 0$, elsewhere.

Remark: This distribution is often called the Weibull distribution. Note that the conditional rate of failure $g(t)$ is

decreasing	if	$0 < k < 1$
constant	if	$k = 1$
increasing	if	$k > 1$.

Example 4: Suppose that $g(t) = \alpha\beta e^{\beta t}$. In this case, $G(t) = \alpha(e^{\beta t}-1)$.

Thus

(26) becomes

$$F(t) = 0 \ , \ t < 0$$
$$= 1 - e^{-\alpha(e^{\beta t}-1)} \ , \ t \geq 0$$

and (27) becomes

$$f(t) = \alpha\beta e^{\beta t} e^{-\alpha(e^{\beta t}-1)} \ , \ t \geq 0 \ .$$

Remark: This kind of distribution occurs in extreme value theory [16].

It should be noted that there is a great deal of similarity between formulae (14) and (26). Actually, these two models have a great deal in common, the difference being only that in one model we make specific assumptions regarding an underlying Poisson process which generates dangerous peaks, whereas in the second model the conditional rate of failure function, $g(t)$, is made central to the discussion. $g(t)$ depends on the environment,

on the items under test, and on the interaction of items being tested and the environment. If we try to make explicit the dependence of $g(t)$ on the environment, we will be led back, essentially, to the first model. It is also interesting to note that we are led to distributions which are in a sense rather direct generalizations of the simple exponential distribution. This is very important because this makes it possible to change the time scale in such a way that all theoretical results obtained under a purely exponential hypothesis are valid for the more general situation. Thus, in the second model, if t represents the time to failure, then $u = G(t) = \int_0^t g(\tau)d\tau$ is distributed with c.d.f. $1 - e^{-u}$, $u \geq 0$, and with p.d.f. e^{-u}, $u \geq 0$. As a consequence of this, many of the theorems, tests, estimation procedures, formulae, etc., become valid in a much more general situation if one replaces failure times t_i by generalized times, $u_i = G(t_i)$.

Another interesting feature of the models is that they lead to all kinds of useful distributions, some of which arise in other connections. As examples we mention the so-called Weibull distributions and extreme value distributions.

4. Two Other Failure Models

We now consider briefly two other models. In the first of these, let us assume that a device subject to an environment E will fail when exactly $k \geq 1$ shocks occur and not before. If shocks occur at a Poisson rate λ, then the waiting time T_k (or life) until the item fails is described by the p.d.f.

$$(28) \qquad f_k(t) = \lambda^k t^{k-1} e^{-\lambda t}/(k-1)! \quad , \quad t \geq 0$$
$$= 0 \text{ , elsewhere}$$

and by the c.d.f.

$$(29) \qquad F_k(t) = \Pr(T_k \leq t) = \sum_{j=k}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \quad .$$

The derivations of (28) and (29) are simple. Thus to get $f_k(t)$ we note that

$$(30) \qquad \Pr(t < T_k < t + \Delta t) = \Pr[\text{exactly } (k-1) \text{ shocks in } (0,t) \text{ and } 1 \text{ shock in } (t,t+\Delta t)]$$
$$= \frac{(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!} \lambda \Delta t$$

$$\therefore f(t) = \lambda^k t^{k-1} e^{-\lambda t}/(k-1)! \quad , \quad t \geq 0 \ .$$

To derive $F_k(t)$ we note that

$$(31) \qquad \Pr(T_k > t) = \Pr[k-1 \text{ or fewer failures in } (0,t)]$$
$$= \sum_{j=0}^{k-1} e^{-\lambda t}(\lambda t)^j/j! \quad .$$

Therefore,

$$(32) \qquad F_k(t) = \Pr(T_k \leq t) = 1 - \Pr(T_k > t)$$
$$= 1 - \sum_{j=0}^{k-1} e^{-\lambda t}(\lambda t)^j/j! = \sum_{j=k}^{\infty} e^{-\lambda t}(\lambda t)^j/j! \quad .$$

For the special case where $k = 1$, (28) and (29) become (6) and (7), respectively.

It is quite clear that formulae (8), (9), (10), (11), (14), (15), and (19) can all be suitably generalized to the case where the life time is given by T_k and not by T_1. Thus, for example, the analogues of (14) and (15) become

$$(33) \qquad \Pr(T_k \leq t) = 1 - \sum_{j=0}^{k-1} e^{-\lambda P(t)}[\lambda P(t)]^j/j! = \sum_{j=k}^{\infty} e^{-\lambda P(t)}[\lambda P(t)]^j/j!$$

and

$$(34) \qquad f_k(t) = [\lambda P(t)]^{k-1}\, e^{-\lambda P(t)}\, \lambda p(t)/(k-1)!$$

In essence, one can say that this simple model gives rise to the type III distribution and its generalizations.

We have seen in our earlier models that under certain assumptions one gets the Weibull and extreme value distributions as possible life distributions. We should now like to examine the relevance of an extreme value model. It is our feeling that what we are saying applies to failure problems involving corrosion. Suppose, for example, that corrosion is essentially a pitting phenomenon and that failure is associated with perforation at the deepest pit. The time to failure may then be viewed as follows: let t_i be the time required to perforate the material at the i^{th} pit. Then the time to failure is $\min \{t_i\}$. If the pit depths follow an exponential distribution, and if we assume that time to perforation at a pit is linearly dependent on the thickness of the coating minus the pit depth, then extreme value theory would

lead one to expect a life time distribution of the form $A e^t e^{-Ae^t}$ [2,8]. For further details on extreme values in this connection one should see papers by Epstein [9,10], Epstein and Brooks [11], and Gumbel [15,16].

Remark 1: There have recently appeared results in the literature that failures of complex mechanisms tend to be exponentially distributed. This has a theoretical justification since the times between failures of the complex mechanism (we are assuming that it is repaired after it breaks down and put back into service) result from a superposition of the failure patterns of the parts making up the mechanism. It has been shown by D. R. Cox and W. L. Smith [5,6] that to a good approximation this kind of superposition gives rise to an exponential distribution of times between successive breakdowns.

Remark 2: It will be noted in the foregoing that none of the failure models led to a normal or logarithmic normal distribution of life. How, then, does one explain the fact that some observed life time distributions appear to be normal or logarithmic-normal? It seems to us that in our considerations we have assumed that in one way or another sudden shocks in the environment were important. If this is the case, then one must be led to the exponential distribution or suitable generalizations of it. But if failure is caused by a wear-out mechanism or is a consequence of accumulated wear, then we assert that the normal distribution can be expected. Thus, if an accumulation of k failures is required for failure, we have been led to a type III distribution which will tend to normality for k large.

Another possibility is that failure occurs after an essential substance has been used up. In this case, the time to failure might be proportional to the amount of this substance in the particular specimens being tested. If the amount of the substance varies from specimen to specimen according to a normal distribution, then one would get a normal distribution of life times.

What about logarithmic normal distributions? It seems to us that such a distribution can arise in either of two ways:

(i) as an approximate fit to skewed distributions like the Weibull or type III, or

(ii) if failure depends on using up some critical substance, the amount of which varies according to a logarithmic normal distribution from specimen to specimen.

It may be noted that the logarithmic normal distribution of lives seems to occur particularly in some biological problems. Whether this results in accordance with (i) or (ii) is generally not clear.

Remark 3: It is interesting to note that the failure distribution given by (34) is almost identical with the one given by Z.W. Birnbaum and S. S. Saunders [3] in their recent paper in which they give a statistical model for life-length of structures under dynamic loading (i.e., fatigue). One difference is that they make the conditional rate of failure function central to their discussion while we make more explicit use of the underlying Poisson process generating dangerous peaks. The other difference is that ~~they are~~ dealing with a multi-component structure, which initially has m

components. They assume further that, in the course of time, one component after another fails and that there is a critical number of failures $k \leq m$, such that the entire structure fails when k of its components fail. The analogue of their Assumption B, if one uses the Poisson model, is the following: if a peak in the stochastic process (the peaks are assumed to occur at rate λ) occurs at time t and if j of the components in the structure have failed prior to time t, then each of the remaining $(m-j)$ components has a conditional probability of failure given by $p(t)/(m-j)$. It is then very easy to show that the p.d.f. of S_k, the life of structure, is given by (34) and further that $2\lambda \int_0^{S_k} p(\tau)d\tau$ is distributed as $\chi^2(2k)$. This is an analogue of the theorem given on p. 154 of [3].

Remark 4: We have shown in this paper that a possible theory of failure is based on the Poisson process. A generalization of Poisson processes is given by "birth and death" processes. It has recently been shown in a note by Weiss [17] that some kinds of mechanical failure, such as creep failure of oriented polymeric filaments under tensile stresses, (see Coleman [4]) can be viewed as "pure death" processes. Essentially the theory assumes that there are initially N_0 fibers, each of which, independent of the other fibers, is subject to failure under load. Failure occurs when all the fibers have failed. Two special cases are considered:

(i) the probability of a single fiber failing in the time interval $(t, t+\Delta t)$ is given by $\psi(t)\Delta t$;

(ii) the probability of a single fiber failing in $(t, t+\Delta t)$ when $n(\leq N_0)$ fibers are left is given by $\frac{N_0}{n}\psi(t)\Delta t$.

A "pure death" process where the probability of a fiber failing is given by (i) results in the following life destribution:

(35) $\quad \Pr(T \leq t) = \Pr(0 \text{ fibers survive time } t)$

$$= [1-e^{-\int_0^t \psi(\tau)d\tau}]^{N_o}$$

The p.d.f. of T is given by

(36) $$f(t) = N_o \psi(t)\, e^{-\int_0^t \psi(\tau)d\tau} [1-e^{-\int_0^t \psi(\tau)d\tau}]^{N_o-1} \quad .$$

In the special case where $\psi(\tau) \equiv \lambda$, $\tau > 0$, then (35) becomes

(35') $$\Pr(T \leq t) = [1-e^{-\lambda t}]^{N_o} \quad ,$$

and (36) becomes

(36') $$f(t) = N_o \lambda e^{-\lambda t}[1-e^{-\lambda t}]^{N_o-1} \quad .$$

Furthermore

(37) $$E_\lambda(T) = \lambda \; [1 + \tfrac{1}{2} + \cdots + \tfrac{1}{N_o}] \sim \frac{\log N_o}{\lambda} \quad .$$

A "pure death" process where the probability of a fiber failing is given by (ii) results in the life distribution:

(38) $\quad \Pr(T \leq t) = \Pr(0 \text{ fibers survive time } t)$

$$= 1 - \sum_{j=0}^{N_o-1} [\int_0^t N_o \psi(\tau)d\tau]^j \; \frac{e^{-\int_0^t N_o \psi(\tau)d\tau}}{j!} \quad .$$

The p.d.f. of T is given by

$$f(t) = [\int_0^t N_o \psi(\tau)d\tau]^{N_o-1} e^{-\int_0^t N_o \psi(\tau)d\tau} N_o \psi(t)/(N_o-1)! \quad . \tag{39}$$

In particular if $\psi(\tau) \equiv \lambda$, $\tau \geq 0$, then (38) and (39) become

$$\Pr(T \leq t) = 1 - \sum_{j=0}^{N_o-1} e^{-N_o\lambda t} (N_o\lambda t)^j/j! \tag{38'}$$

and

$$f(t) = N_o\lambda(N_o\lambda t)^{N_o-1} \quad e^{-N_o\lambda t}/(N_o-1)! \quad . \tag{39'}$$

Further

$$E_\lambda(T) = \frac{1}{\lambda} \quad . \tag{40}$$

It should be noted that assumption (ii) in the Weiss model is essentially the same as assumption B in the Birnbaum - Saunders paper and hence the p.d.f.'s of life which result must agree. It should also be noted that (36) is the p.d.f. of the largest value in a sample of size N_o drawn from a distribution with c.d.f. $1 - e^{-\int_0^t \psi(\tau)d\tau}$, while (39) is a Type III distribution.

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